**Time Development in Heisenberg Picture**

There are two equivalent ‘pictures’ regarding time-development, in quantum mechanics. One is the Schrodinger picture, in which the state vector is considered to evolve with time. The other is the Heisenberg picture, in which the operators are rather considered to evolve with time. They are equivalent, and distinguished only in the sense that in the Schrodinger picture, you apply U(t) to the state vectors, whereas in the Heisenberg picture you apply it to the operators.

Now we’ll consider the Heisenberg Picture. The easiest way to figure out how opeators develop is by using the fact that since the two pictures are equivalent, they must give the same expectations. For instance suppose we have a particle whose wavevector is changing with time |ψ(t)> and we wish to calculate the expectation of some operator Â as a function of time. Then we would have:



which could be written:



which implies that the time developed operator is:



The subscript H is there to indicate ‘Heisenberg’. The combination of those three operators is always given the subscript H for shorthand, and is considered the time-development of the operator Â. We would just write Â(t) to indicate the time development of Â, but unfortunately that notation has been hijacked a little by our time-dependent perturbations, which we have written as, for instance:



This is just a time-*dependent* operator, but this should not be confused with the time-developed operator, , which is:



Observe that this illustrates the basic difference between the Schrodinger picture and the Heisenberg picture. Namely, when you make U(t) operate on |ψ0> then you’re implicitly using the ‘Schrodinger’ picture. When you make the U(t)’s operate on the Â, then you’re implicitly using the ‘Heisenberg’ picture. But it’s not like you have to stay in one picture or another throughout a calculation. That this expression is accurate can be confirmed by verifying that it does lead to the expected equations of motion for Â in the Heisenberg picture. For instance,



And so we can write,



where for the last term in the expression, the derivative is before the Heisenberg representation is formed. As we can see, this is precisely the time-development postulated by the Heisenberg picture, so this is the correct time development. Note finally that we can write this as:



which is often a more convenient formula to work out the time-development. Let us examine a few special cases below.

(1) Â itself isn’t time-dependent, in which case we have



(2) And Ĥ commutes with itself at different times



We’ll notice that the time development of our operator A, follows the same equation as its counterpart in Classical Mechanics, just replacing the Poisson bracket with the commutator, and a factor of i. And in fact this is often taken to be the fundamental postulate of time-development, from which we could go backwards and derive the form of U(t), and then also the Schrodinger equation for |ψ(t)>.

It is suggested above, and we’ll see evidence of this below. That the quantum mechanical operators obey classical equations of motion. And this is in fact the case. To put this in terms that will be useful for when we do QFT, we’ll say that equations of motion of the operators follow from minimizing the classical action. Say we have:



Then, minimizing the action we have:



Sort of unrelated note…If is an analytic function of and , say, then it follows that:



This is certainly true for any power law function:



and so it is true for any analytic function of these variables since it can be reduced to a sum of powers law functions by Taylor series.

**Example**

Let’s consider a general force field in 1D. Let’s show that the time-development of the momentum and position operators follow their expected classical laws. Leaving off H subscript.



and similarly we’ll find with x.



which is true classically as well. Note that in general we may say:



**Harmonic Operators**

Just as harmonically developing wavefunctions contain important information about the energy levels of the system, so too do harmonically developing operators. For instance, consider an operator, say q, which evolves with time according to:



Then it follows that if |n> is an eigenstate of H with eigenvalue En, then q0|n> is an eigenstate of H with eigenvalue En ± ω0. So in effect it is a sort of creation/annihilation operator, and ω0 is a system *excitation*. For consider a time-independent H and the equation for q evolution:



which implies that q0|n> is an eigenstate of H since,



and so this is proved. Additionally, if +ω0, then it’s a creation operator apparently, and if -ω0, then an annihilation operator. This would prove a much easier method of analyzing the HO for example, than the Schrodinger equation. And it is an illustration how finding the time dependence of the operators can aid in finding the eigenvalues. More generally, the time development of any operator will tell us the energy eigenvalues. Consider:



and expand the operator in the eigenbasis,



and so solving for the matrix elements we have,



and the FT is,



and so the temporal Fourier transform of the q(t) gives us the system excitations, consistent with our earlier analysis of the special case of the creation/annihilation operator.

**Explicit expression for AH(t)**

The ease with which we calculated the xH(t) above came from the fact that we were able to exactly solve the classical equations of motion to get . But sometimes, most times, this isn’t possible, and so we need some perturbative way to calculate ÂH(t). We can do this in the following way. Start with the Heisenberg equations of motion. And assume for simplicity that Â is a time-independent operator (they always are in practice). Then we have:



Now integrate both sides from 0 to t to get:



Now we iterate this solution by writing,



and iterate again to get:



Continuing the process, we see that the solution is:



And if we were feeling especially liberal with the notation, we might even write:



where T is the time-ordering operator (orders the H’s so that those with later times are further to the left) As a special case, consider a time-independent H. Then we have:



We can write this in a slightly suggestive form,



This is meant to evoke the idea of a Taylor series expansion, but with repeated commutation replacing the repeated differentiation. This is fruitful analogy because commutation plays a role very like that of differentiation.